Int. J. Heat Mass Transfer. Vol. 10, pp. 1757–1766. Pergamon Press Ltd. 1967. Printed in Great Britain

FLOW AND HEAT TRANSFER IN THE AXISYMMETRIC BOUNDARY LAYER OVER A CIRCULAR CYLINDER

SIAVASH ESHGHY and ROBERT W. HORNBECK
Carnegie–Mellon University, Pittsburgh, Pennsylvania 15213

(*Received* 24 *December* 1966 *and in revised form* 23 *June* 1967)

Abstract—The problem of laminar boundary-layer flow and heat transfer over a long thin cylinder in uniform flow has been analyzed. Solutions are obtained for small as well as large values of the curvature parameter. Solutions valid for small values of curvature are extended to apply to in ermediate values of curvature as well. This is done by casting the expressions into fractions. Finally, these fractions are joined with the asymptotic solution to produce heat-transfer and skin friction results for all values of the curvature parameter. The shapes of the velocity and temperature profiles are given graphically.

NOMENCLATURE

- a, see equation (65) and Table 2;
- A_n, see equation (60);
- b, see equation (65) and Table 2;
- b_n, see equation (39);
- B_n, see equation (38);
- c, see equation (65) and Table 2;
- c_n, see equation (39);
- C_n, see equation (38);
- f_n, see equation (10);
- F, asymptotic stream function;
- F_n, see equation (30);
- g_n, see equation (11);
- $g(\sigma)$, see equations (54, 55) and Table 2;
- k, thermal conductivity;
- m, an integer;
- n, an integer;
- q, local heat flux;
- q_0, local heat flux for zero curvature;
- $\bar{q}$, average heat flux $= 1/x \int_0^x q(x)\,\mathrm{d}x$;
- $\bar{q}_0$, average heat flux for zero curvature;
- r, distance from the axis of cylinder;
- R, cylinder radius;
- t, dimensionless temperature (asymptotic solution);
- t_n, see equation (31);
- T, fluid temperature;
- T_1, fluid temperature outside the layer;
- T_w, uniform wall temperature;
- u, velocity component in x-direction;
- u^*, dimensionless u, see equation (5);
- U, uniform main stream velocity;
- v, velocity component in r-direction;
- v^*, dimensionless v, see equation (5);
- x, distance along cylinder axis;
- z, dimensionless transverse coordinate, see equation (26).

Greek symbols

- β, see equation (26);
- γ, Euler's constant 0·5772157 . . . ;
- δ, true displacement thickness $(\delta + R)^2 - R^2 = 2\int_R^\infty (1 - u/U)\,r\,\mathrm{d}r$;
- η, dimensionless transverse coordinate, see equation (5);
- θ, dimensionless temperature, see equation (5);
- θ_n, see equation (12);
- μ, fluid viscosity;
- ν, kinematic viscosity;
- ξ, the curvature parameter, see equation (5);
- σ, Prandtl number;
- τ, local skin friction;
- τ_0, local skin friction for zero curvature;
- $\bar{\tau}$, average skin friction;
- $\bar{\tau}_0$, average skin friction for zero curvature.

1. INTRODUCTION

THE PROBLEM of flow in the axial direction over the outer surface of a cylinder represents an example where an ordinarily second-order effect becomes of primary significance. The strong existence of transverse curvature in this flow justifies ignoring all other second order effects such as that of the displacement speed (see for instance Van Dyke [1]).

Extensive work on the flow aspects of this problem may be found in the literature. Seban and Bond [2] obtained a series solution valid for small values of the curvature parameter. The solutions given by Seban and Bond together with the numerical corrections of Kelly [3] constitute the extent of the literature dealing with solutions valid for small values of the curvature parameter $\xi = (\nu x/UR^2)^{\frac{1}{2}}$ where ν is the kinematic viscosity, x the distance along the cylinder, U the velocity of the uniform stream and R the radius of the cylinder. The Seban–Bond–Kelly solution for skin friction which applies to about $\xi = 0{\cdot}2$ already shows a strong influence of ξ on the skin friction.

The flow behaviour for large values of ξ is discussed by Batchelor [4] on the basis of analogy to a transient conduction problem solved by Rayleigh [5]. A more systematic approach is given by Stewartson [6] who expands the flow quantities in terms of descending powers of log $(4\xi^2/D)$ where D is a number whose natural logarithm is the Euler number. Glauert and Lighthill [7] finally resolve the flow problem by first obtaining a fairly accurate solution by integral technique (good for all values of ξ). They then connect the Seban–Bond–Kelly solutions with their own asymptotic solutions. The interpolation is arrived at by inspection, using the integral solution as a guide.

Although a two term expansion for heat-transfer quantities is given by Seban and Bond for a Prandtl number of 0·715, an exact and comprehensive treatment of the temperature problem remains apparently nonexistent.

The present work consists of series solutions obtained for small values of ξ and Prandtl numbers of 0·7, 1 and 10. The method used here is slightly different from that given by Seban and Bond. Furthermore an asymptotic solution is derived for the flow and heat-transfer quantities in a manner similar to one given by Glauert and Lighthill. The method does not require numerical integration and therefore the heat-transfer results may be calculated for any Prandtl number. The small ξ solutions for skin friction and heat transfer are cast into fractions and joined with the asymptotic series at appropriate values of ξ. This leads to fairly simple expressions for skin friction and heat transfer valid for all values of ξ.

Finally a finite difference solution generated up to $\xi = 4$ provides information on the shapes of the velocity and temperature profiles which should be of qualitative interest.

2. FORMULATION OF THE PROBLEM

Designating x as the axis of the circular cylinder and r as the radial coordinate, the governing boundary-layer equations with the inclusion of transverse curvature can be written as,

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial r} = \frac{\nu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right), \tag{1}$$

$$\frac{\partial u}{\partial x} + \frac{v}{r} + \frac{\partial v}{\partial r} = 0, \tag{2}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial r} = \frac{\nu}{\sigma r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right), \tag{3}$$

where u and v are velocity components in x and r directions respectively, T is the fluid temperature and σ the Prandtl number.

The boundary conditions are,

$$\begin{aligned} &u(x, R) = v(x, R) = 0; \qquad u(x, \infty) = U \\ &T(x, R) = T_w; \qquad T(x, \infty) = T_1. \end{aligned} \tag{4}$$

Here T_w is the uniform wall temperature and T_1 the temperature of the main stream.

The above equations assume constant properties, zero pressure gradients and negligible viscous dissipation. The assumption of zero pressure gradient is of course not valid, in general, near the nose region ($x = 0$) but should be a reasonable one a few diameters downstream—as long as no flow separation occurs.

A curvature parameter ξ is now defined which is similar to the ones given by the previous workers, namely a quantity proportional to the ratio of flat plate boundary-layer thickness to the cylinder radius.

$$\xi = \frac{1}{R}\left(\frac{\nu x}{U}\right)^{\frac{1}{2}}$$

The parameter ξ represents the extent of curvature present in the flow but not uniformly for all x since the growth of the layer will in fact be slower than $x^{\frac{1}{2}}$.

3. SOLUTION FOR SMALL VALUES OF ξ

Consider the following transformation of functions and variables

$$\left.\begin{aligned} \xi &= \left(\frac{\nu x}{UR^2}\right)^{\frac{1}{2}} \\ \eta &= (r - R)\left(\frac{U}{\nu x}\right)^{\frac{1}{2}} \\ u^* &= \frac{u}{U} \\ v^* &= 2v\left(\frac{x}{\nu U}\right)^{\frac{1}{2}} \\ \theta &= (T - T_1)/(T_w - T_1). \end{aligned}\right\} \quad (5)$$

Substitution of (5) into (1), (2) and (3) results in the following partial differential equations

$$u^*(\xi u^*_\xi - \eta u^*_\eta) + v^* u^*_\eta = 2\left(u^*_{\eta\eta} + \frac{\xi}{1 + \xi\eta} u^*_\eta\right) \quad (6)$$

$$\xi u^*_\xi - \eta u^*_\eta + v^*_\eta + \frac{\xi}{1 + \xi\eta} v^* = 0 \quad (7)$$

$$u^*(\xi\theta_\xi - \eta\theta_\eta) + v^*\theta_\eta = \frac{2}{\sigma}\left(\theta_{\eta\eta} + \frac{\xi}{1 + \xi\eta}\theta_\eta\right) \quad (8)$$

subject to the boundary conditions,

$$\begin{aligned} &u^*(\xi, 0) = v^*(\xi, 0) = 0; \qquad u^*(\xi, \infty) = 1 \\ &\theta(\xi, 0) = 1; \qquad \theta(\xi, \infty) = 0. \end{aligned} \quad (9)$$

Here subscripts denote differentiation. Assume next the following series representation of velocities and temperature:

$$u^* = f_0'(\eta) + \xi f_1'(\eta) + \xi^2 f_2'(\eta) + \cdots \quad (10)$$

$$v^* = g_0(\eta) + \xi g_1(\eta) + \xi^2 g_2(\eta) + \cdots \quad (11)$$

$$\theta = \theta_0(\eta) + \xi\theta_1(\eta) + \xi^2\theta_2(\eta) + \cdots \quad (12)$$

Equation (7) is satisfied to $0(\xi^3)$ when

$$g_0 = \eta f_0'' - f_0' \quad (13)$$

$$g_1 = \eta f_1'' - 2f_1' - \eta f_0' + 2f_0 \quad (14)$$

$$g_2 = \eta f_2' - 3f_2 - \eta f_1' + 3f_1 + \eta^2 f_0' - 2\eta f_0. \quad (15)$$

Further substitution and rearrangement results in three sets of ordinary differential equations with η as the variable.

$$\left.\begin{aligned} &2f_0^{\text{iv}} + f_0' f_0''' = 0 \\ &(2/\sigma)\theta_0'' + f_0'\theta_0' = 0 \\ &f_0(0) = f_0'(0) = f_0''(0) = 0; \qquad f_0''(\infty) = 1 \\ &\theta_0(0) = 1; \qquad \theta_0(\infty) = 0 \end{aligned}\right\} \quad (16)$$

$$\left.\begin{aligned} &2f_1^{\text{iv}} + f_0' f_1''' - f_0'' f_1'' + 2f_0''' f_1' \\ &\quad + (\eta f_0' - 2f_0 + 2) f_0''' = 0 \\ &(2/\sigma)\theta_1'' + f_0'\theta_1' - f_0''\theta_1 \\ &\quad + 2(f_1' + \eta f_0'/2 - f_0 + 1/\sigma)\theta_0' = 0 \\ &f_1(0) = f_1'(0) = f_1''(0) = 0; \qquad f_1''(\infty) = 0 \\ &\theta_1(0) = 0; \qquad \theta_1(\infty) = 0 \end{aligned}\right\} \quad (17)$$

$$\left.\begin{aligned} &2f_2''' + f_0' f_2'' - 2f_0'' f_2' + 3f_0''' f_2 \\ &\quad + 2\eta f_1^{\text{iv}} + 2(f_1' + \eta f_0' - f_0 + 1) f_1''' \\ &\quad -(f_1'' + \eta f_0'')f_1'' + 3\eta f_0''' f_1' - 3f_0''' f_1 = 0 \\ &(2/\sigma)\theta_2'' + f_0'\theta_2' - 2f_0''\theta_2 + 2\eta\theta_1''/\sigma \\ &\quad + 2(f_1' + \eta f_0' - f_0 + 1/\sigma)\theta_1' \\ &\quad -(f_1'' + \eta f_0'')\theta_1 + 3(f_2 + \eta f_1' - f_1)\theta_0' = 0 \\ &f_2(0) = f_2'(0) = 0; \qquad f_2'(\infty) = 0 \\ &\theta_2(0) = 0; \qquad \theta_2(\infty) = 0. \end{aligned}\right\} \quad (18)$$

It should be noted here that the procedure above is different from that given by Seban and Bond. The difference arises from the choice of η in equations (5) which does not allow an *a priori* definition of a stream function. The only advantage of the above procedure is the convenience in the presentation of the profiles.

Equations (16–18) were solved numerically using a procedure which provided consistent truncation errors of the order of 10^{-7}. Figure 1 shows plots of f_0'', f_1'' and f_2' and Figure 2 shows θ_0, θ_1 and θ_2 plotted for Prandtl numbers 0·7 and 10. For Prandtl number of unity θ's are directly related to f's as

$$\theta_0 = 1 - f_0''$$
$$\theta_1 = - f_1''$$
$$\theta_2 = - f_2'$$

which may be deduced from the analogy between velocity and temperature distributions

$$\theta = 1 - u^*, \qquad \sigma = 1. \tag{19}$$

The initial conditions resulting from the solution of (16–18) are listed in Table 1. According to Kelly [3], $f_1'''(0)$ and $f_2''(0)$ are 0·697 and −0·638 respectively. The latter was quoted incorrectly as −0·797 in reference [7]. The initial condition for Blausius profile is given with eight places which may be of value for future work (a more accurate procedure was used here).

The skin friction τ and the heat flux q may now be expressed as,

$$\frac{\tau R}{\mu U} = \frac{1}{\xi}\{f_0'''(0) + \xi f_1'''(0) + \xi^2 f_2''(0) + \cdots\} \tag{20}$$

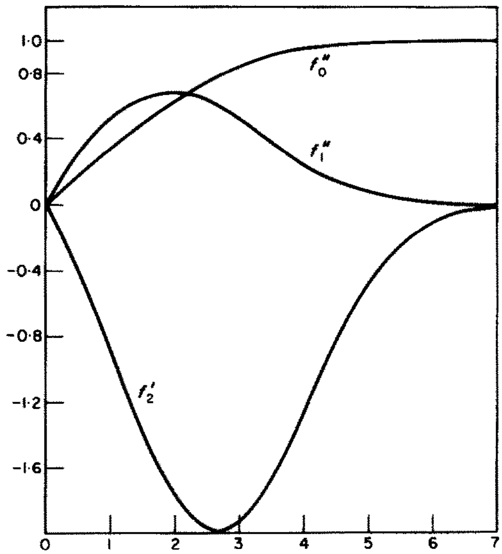

FIG. 1. Velocity functions.

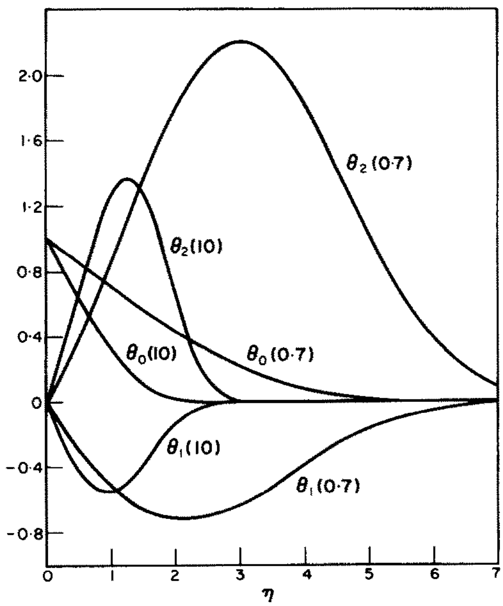

FIG. 2. Temperature functions.

Table 1

σ	$f_0'''(0)$	$f_1'''(0)$	$f_2''(0)$	$\theta_0'(0)$	$\theta_1'(0)$	$\theta_2'(0)$
0·7	0·33205733	0·69432	−0·65658	−0·29268	−0·66364	0·62688
1	0·33205733	0·69432	−0·65658	−0·33206	−0·69432	0·65658
10	0·33205733	0·69432	−0·65658	−0·72814	−1·01761	1·12404

$$\frac{qR}{k(T_w - T_1)} = -\frac{1}{\xi}\{\theta_0'(0) + \xi\theta_1'(0) + \xi^2\theta_2'(0) + \cdots\}. \quad (21)$$

Here μ is the viscosity and k the thermal conductivity of the fluid. If these quantities are now normalized with respect to their corresponding flat plate values (zero curvature), the following expressions result

$$\tau/\tau_0 = 1 + [f_1'''(0)/f_0'''(0)]\,\xi + [f_2''(0)/f_0'''(0)]\,\xi^2 + \cdots. \quad (22)$$

$$q/q_0 = 1 + [\theta_1'(0)/\theta_0'(0)]\,\xi + [\theta_2'(0)/\theta_0'(0)]\,\xi^2 + \cdots. \quad (23)$$

The right of (22) is only a function of ξ but the right of (23) is a function of σ as well as ξ. The other quantity of interest is the true displacement thickness δ as defined by Kelly.

$$\delta/R = \xi\{1{\cdot}7208 - 1{\cdot}4088\xi + 4{\cdot}1052\xi^2 + \cdots\}. \quad (24)$$

The limits which lead to (24) are as follows:

$$\left.\begin{aligned} &\lim_{\eta\to\infty}(\eta - f_0') = 1{\cdot}7207876 \\ &\lim_{\eta\to\infty}(f_1' - \eta^2/2 + \eta f_0' - f_0) = -0{\cdot}071716 \\ &\lim_{\eta\to\infty}(f_2 + \eta f_1' - f_1) = -1{\cdot}6809. \end{aligned}\right\} \quad (25)$$

4. THE ASYMPTOTIC SOLUTION

Now we seek solutions for very large values of ξ where the thicknesses of the layers are many times greater than the cylinder radius. For large values of ξ, the use of transformation (5) is inconvenient and therefore a different set of functions and variables are introduced.

$$\left.\begin{aligned} &\beta = \log(4\xi^2) \\ &z = \frac{Ur^2}{4\nu x} \\ &u = \tfrac{1}{2}U\frac{\partial F}{\partial z} \\ &v = \frac{\nu}{r}\left(z\frac{\partial F}{\partial z} - F - \frac{\partial F}{\partial \beta}\right) \\ &t = 2\left(1 - \frac{T - T_1}{T_w - T_1}\right). \end{aligned}\right\} \quad (26)$$

The expression for v in (26) is chosen such that (2) is identically satisfied. Further substitution reduces (1) and (3) to the following forms:

$$zF_{zzz} + (1 + \tfrac{1}{2}F + \tfrac{1}{2}F_\beta)F_{zz} - \tfrac{1}{2}F_zF_{z\beta} = 0 \quad (27)$$

$$(1/\sigma)zt_{zz} + (1/\sigma + \tfrac{1}{2}F + \tfrac{1}{2}F_\beta)t_z - \tfrac{1}{2}F_zt_\beta = 0 \quad (28)$$

Subject to the boundary conditions

$$\left.\begin{aligned} &F(e^{-\beta},\beta) = F_z(e^{-\beta},\beta) = F_\beta(e^{-\beta},\beta) = 0 \\ &F_z(\infty,\beta) = 2 \\ &t(e^{-\beta},\beta) = 0; \qquad t(\infty,\beta) = 2 \end{aligned}\right\} \quad (29)$$

We will now consider an expansion of the form

$$F(z,\beta) = F_0(z) + \beta^{-1}F_1(z) + \beta^{-2}F_2(z) + \cdots \quad (30)$$

$$t(z,\beta) = t_0(z) + \beta^{-1}t_1(z) + \beta^{-2}t_2(z) + \cdots. \quad (31)$$

Substitution of (30) and (31) in (27) and (28), and neglecting terms of the order of β^{-3} results in the following set of ordinary differential equations:

$$zF_0''' + (1 + \tfrac{1}{2}F_0)F_0'' = 0 \quad (32)$$

$$(1/\sigma)zt''_0 + (1/\sigma + \tfrac{1}{2}F_0)t_0' = 0 \quad (33)$$

$$zF_1''' + (1 + \tfrac{1}{2}F_0)F_1'' + \tfrac{1}{2}F_0''F_1 = 0 \quad (34)$$

$$(1/\sigma)zt_1'' + (1/\sigma + \tfrac{1}{2}F_0)t_1' + \tfrac{1}{2}F_1t_0' = 0 \quad (35)$$

$$zF_2''' + (1 + \tfrac{1}{2}F_0)F_2'' + \tfrac{1}{2}F_0''F_2 + \tfrac{1}{2}F_1F_1'' - \tfrac{1}{2}F_1F_0'' + \tfrac{1}{2}F_1'F_0' = 0 \quad (36)$$

$$(1/\sigma)zt_2'' + (1/\sigma + \tfrac{1}{2}F_0)t_2' + \tfrac{1}{2}F_1t_1' + \tfrac{1}{2}F_2t_0' + \tfrac{1}{2}F_0't_1 - \tfrac{1}{2}F_1t_0' = 0. \quad (37)$$

Inspection of boundary conditions (29) reveals that these cannot be satisfied term by term and at the same time each term of the series be a function of z alone. The boundary conditions will then be only approximately satisfied with an error of the order of $e^{-\beta}$ as follows. Referring to equations (32–37) and considering that near the wall ($z \to e^{-\beta}$) F_0, F_1, F_2, etc., all approach zero, we can see that F'_n and t_n are all of the form

$$F'_n \sim B_n + C_n \log z \quad (38)$$

$$t_n \sim b_n + c_n \log z, \qquad n = 0, 1, 2, \ldots \quad (39)$$

Relations may be found among the coefficients b_n, c_n, B_n and C_n by satisfying at $z = e^{-\beta}$ the conditions

$$\sum_{n=0}^{\infty} F'_n \beta^{-n} = 0 \quad (40)$$

$$\sum_{n=0}^{\infty} t_n \beta^{-n} = 0. \quad (41)$$

Equation (40) together with (38) leads to the following relations:

$$C_0 = 0; \qquad C_n = B_{n-1}. \quad (42)$$

Similarly (41) and (39) require that

$$c_0 = 0; \qquad c_n = b_{n-1}. \quad (43)$$

Solution of (32) subject to the boundary condition $F'_0(\infty) = 2$ is simply,

$$F'_0 = B_0 = 2; \qquad F_0 = 2z \quad (44)$$

and similarly,

$$t_0 = b_0 = 2. \quad (45)$$

Having satisfied (40) and (41) near the wall, we only have to satisfy conditions for $z \to \infty$ on each of F'_n and t_n to be zero. For $n \geqslant 1$ the differential equations are linear and may be solved analytically in terms of an arbitrary constant. These constants are then determined by comparing (38) and (39) with the limits of these solutions as $z \to 0$. These solutions are as follows

$$F'_1 = 2Ei(-z) \quad (46)$$

$$F_1 = 2[zEi(-z) + e^{-z} - 1] \quad (47)$$

$$F'_2 = 2e^{-z}Ei(-z) - 4Ei(-2z) - [Ei(-z)]^2 + 4Ei(-z)\log z - 6EI(-z) + 2(1 + 3\gamma) \times Ei(-z) \quad (48)$$

$$t_1 = 2Ei(-\sigma z) \quad (49)$$

$$\begin{aligned} t_2 = {} & -2(\sigma + 2)\,EI(-\sigma z) - 2(1 + \sigma) \\ & \times Ei[-(1 + \sigma)z] + 2e^{-\sigma z}\,Ei(-z) \\ & - 2\sigma Ei(-z)\,Ei(-\sigma z) + 2(1 + \sigma)\,Ei(-\sigma z)\log z \\ & + 2[2\gamma + 2\log\sigma + (1 + \gamma)\,\sigma]\,Ei(-\sigma z) \\ & + 2\sigma \int_{\infty}^{z} Ei(-\sigma\lambda)\frac{e^{-\lambda}}{\lambda}\,d\lambda \end{aligned} \quad (50)$$

where,

$$Ei(-z) = \int_{\infty}^{z} \frac{e^{-\lambda}}{\lambda}\,d\lambda$$

$$EI(-z) = \int_{\infty}^{z} \frac{Ei(-\lambda)}{\lambda}\,d\lambda$$

and $\gamma = 0{\cdot}5772157\ldots$.

In deriving expressions (47–50) the following limits were used for the functions involved as $z \to 0$,

$$Ei(-z) \sim \gamma + \log z \quad (51)$$

$$EI(-z) \sim \frac{\pi^2}{12} + \tfrac{1}{2}(\gamma + \log z)^2 \quad (52)$$

$$\int_{\infty}^{z} Ei(-\sigma\lambda)\frac{e^{-\lambda}}{\lambda}\,d\lambda \sim (\gamma + \log z) \times \left[\frac{\gamma}{2} + \log(\sigma z^{\frac{1}{2}})\right] - \frac{\pi^2}{12} - g(\sigma) \quad (53)$$

where

$$g(\sigma) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}\sigma^n, \qquad \sigma \leqslant 1 \quad (54)$$

$$g(\sigma) = -\tfrac{1}{2}(\log\sigma)^2 - \frac{\pi^2}{6} - \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\sigma^{-n}, \qquad \sigma \geqslant 1. \quad (55)$$

The above procedure uniquely determines the coefficients C_n, B_n, c_n and b_n as follows

$$\left.\begin{aligned} &C_0 = 0 \\ &B_0 = C_1 = 2 \\ &B_1 = C_2 = 2\gamma \\ &B_2 = C_3 = 2\gamma^2 - \frac{\pi^2}{2} - 4\log 2 \\ &c_0 = 0 \\ &b_0 = c_1 = 2 \\ &b_1 = c_2 = 2(\gamma + \log\sigma) \\ &b_2 = c_3 = 2\gamma^2 - (1+\sigma)\frac{\pi^2}{3} \\ &\quad -2(1+\sigma)\log(1+\sigma) + (2-\sigma) \\ &\times (\log\sigma)^2 + 2(2\gamma+\sigma)\log\sigma - 2\sigma g(\sigma). \end{aligned}\right\} \quad (56)$$

The skin friction and heat transfer may now be expressed as

$$\frac{\tau R}{\mu U} = \sum_{n=1}^{\infty} C_n \beta^{-n} \quad (57)$$

$$\frac{qR}{k(T_w - T_1)} = \sum_{n=1}^{\infty} c_n \beta^{-n}. \quad (58)$$

Expression (54) for $g(\sigma)$ was arrived at by direct manipulation of the integral on the left-hand side of (53). Expression (55) results if the same sort of manipulation follows integration by parts. Before (55) was deduced, we attempted to extend the convergence of series (54) to values of $\sigma > 1$. Series (54) was transformed to the following asymptotic series

$$g(\sigma) \sim \sum_{n=1}^{N} A_n \left(\frac{\sigma}{1+\sigma}\right)^n + \mathscr{R}_N \quad (59)$$

where

$$A_n = (n-1)! \sum_{m=1}^{n} \frac{(-1)^m}{m^2[(n-m)!][(m-1)!]} \quad (60)$$

and $\mathscr{R}_N$ is the remainder of the series.

Equation (59) as an infinite series is of course divergent for all $\sigma \neq 0$, however, when truncated it gives reasonable approximations to $g(\sigma)$ as long as σ is not too large. It turned out nevertheless that the accuracy for $\sigma = 10$ was not adequate. For instance (59) with fourteen terms gives the values of $-0{\cdot}605$, $-0{\cdot}822$ and $-3{\cdot}73$ for $\sigma = 0{\cdot}7$, 1 and 10 respectively—the exact values being $-0{\cdot}605$, $-0{\cdot}822$ and $-4{\cdot}20$ to three significant digits.

5. JOINING OF THE TWO SOLUTIONS

Solutions given in Section 3 are limited to very small values of ξ and those given in Section 4 are only valid for very large values ξ. We will attempt here to provide information for the intermediate values of ξ by utilizing a matching technique. Although, strictly speaking, joining of profiles is possible, it involves a tedious process leading to expressions too lengthy to be of much value. We will therefore confine this section to joining of skin friction and heat transfer.

We will start by rewriting expression (22) for the normalized skin friction which is independent of Prandtl number

$$\tau/\tau_0 = 1 + \xi\{2{\cdot}09096 - 1{\cdot}97731\,\xi + \cdots\}. \quad (61)$$

We then cast the bracket in (61), using Euler's transformation, into the form

$$\tau/\tau_0 = 1 + \xi\left\{2{\cdot}09096 - 1{\cdot}97731\frac{\xi}{1+\xi} + \cdots\right\}. \quad (62)$$

If the first few coefficients of the series (62) differ from 2, in absolute value, only slightly (a pure conjecture at this point) then for small and intermediate values of ξ, (62) may be approximated by

$$\tau/\tau_0 \sim \frac{1 + 4\xi + 2\xi^2}{1 + 2\xi}. \quad (63)$$

Expression (63) is now used only as a guide for casting (61) into a fraction of the form

$$\tau/\tau_0 = \frac{1 + a\xi + b\xi^2 - c\xi^3}{1 + 2\xi}. \quad (64)$$

Coefficients a and b are then determined from (61) and c is to be found by matching with the asymptotic expression at an appropriate value of $\xi = \xi_m$. We use the same procedure for the heat flux.

$$q/q_0 = \frac{1 + a\xi + b\xi^2 - c\xi^3}{1 + 2\xi}, \qquad \xi \leqslant \xi_m \qquad (65)$$

Table 2 shows the values of a, b, c, ξ_m and also $g(\sigma)$ for various values of σ. The values of the coefficients for skin friction are identical to those for heat flux for Prandtl number of unity.

The foregoing procedure, we feel, is self consistent. The small values obtained for c coefficients indicate that the extrapolation of the small $-\xi$ solution by means of fractions is a reasonable one. Indeed expression (65) with c set equal to zero predicts the heat flux ratios with errors of less than 30 per cent. Although no matching of the slopes was imposed, the approach of the two solutions is rather smooth. An alternative method of joining would have been to match the slopes at $\xi = \xi_m$ and thus determine the coefficient multiplying ξ in the denominator of fraction (65)—this was taken to be 2 based on fraction (63). The derivative of the asymptotic heat flux ratio with respect to ξ, however, is given with much less accuracy than the function itself. The maximum error of the joined solutions should occur somewhere near $\xi = \xi_m$. Equation (65) would therefore slightly underestimate the heat flux for Prandtl numbers 0·7 and 1; for Prandtl number of 10, however, (65) would slightly overestimate the heat flux near $\xi = \xi_m$ because of its slight change of curvature.

Figure 3 shows a plot of the local heat flux ratio for two Prandtl numbers. The ratio of average heat flux is also given in Fig. 4. The heat flux ratios for Prandtl number of unity which are identical to those of the skin friction would plot just below the curves for $\sigma = 0{\cdot}7$. To give an idea of the extent of curvature present in the flow, the values of δ/R are also indicated in Figs. 3 and 4. It is interesting to note that equation (65) with its simple form is probably all that one would ever need in a practical case.

6. THE PROFILES

The small $-\xi$ solution provides expressions for temperature and velocity profiles. These expressions, however, are not valid beyond $\xi = 0{\cdot}05$. The reason is that the rate of convergence of series (10–12) is a function of η. Although convergence near the wall and near the edge of the layer is fairly good, in the middle of the layer it is indeed poor. Extrapolation of the profiles by casting the series into fractions demonstrates the same trend. The asymptotic solution also provides expressions for the profiles but only in terms of z. Although the asymptotic expressions may be written in terms of η, the evaluation of the asymptotic profiles would require laborious numerical evaluation of the integrals involved.

In view of these difficulties, the most convenient method for finding the profiles was felt to be a finite difference technique. Hence, equations (6–9) were written in finite difference forms with initial conditions (at $\xi = 0{\cdot}01$) specified by equations (10–12). The mesh structure was uniform in the ξ-direction ($\Delta\xi = 0{\cdot}002$) but in the η-direction $\Delta\eta = 0{\cdot}025$ from $\eta = 0$ to $\eta = 1$ and $\Delta\eta = 0{\cdot}2$ from $\eta = 1$ to $\eta = 7$. The value of 7 for η was used as the effective infinity. The program was run up to $\xi = 4$ for Prandtl

Table 2

σ	a	b	c	ξ_m	$-g(\sigma)$
0·7	4·26746	2·39306	0·06130	9	0·60516
1	4·09096	2·20462	0·05164	10	0·82247 = $\pi^2/12$
10	3·39755	1·25138	0·01650	20	4·19828

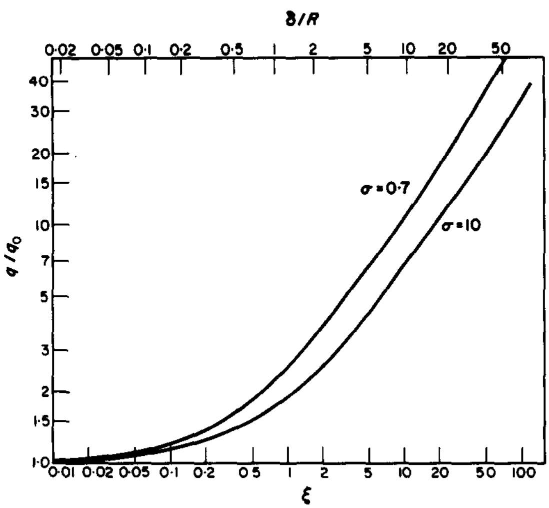

FIG. 3. Local heat flux ratio.

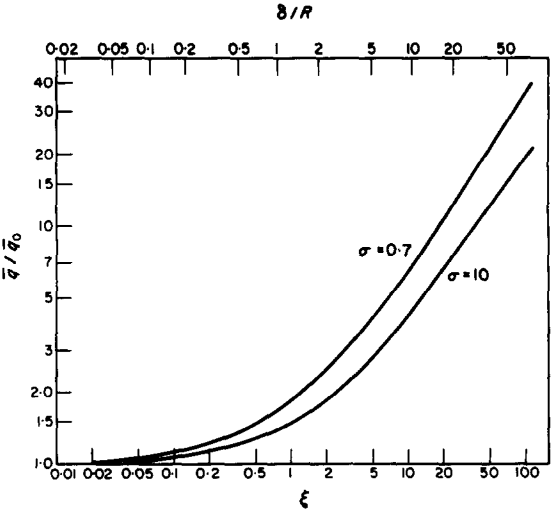

FIG. 4. Average heat flux ratio.

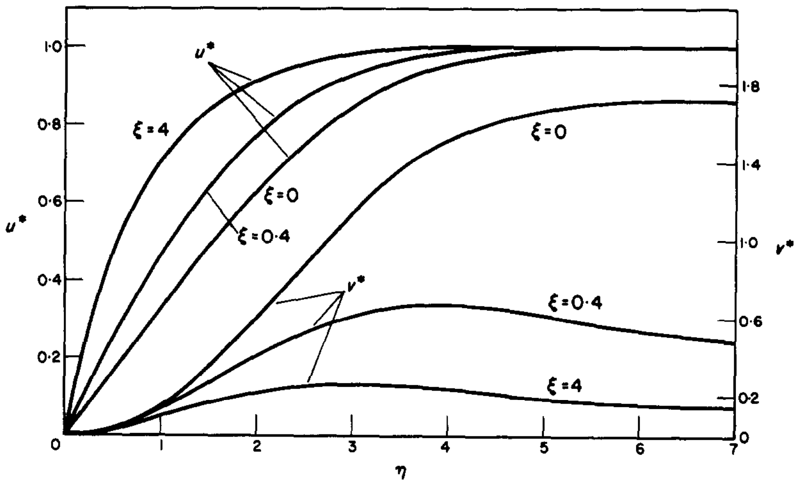

FIG. 5. Velocity profiles.

numbers of 0·7 and 10. The velocity profiles are given in Fig. 5 for three values of ξ. The curves $\xi = 0$ correspond to the Blausius profiles (no curvature). The effect of transverse curvature on u^* is, as expected, a reduction of the boundary-layer thickness. Higher values of ξ would simply shift the u^* profiles more and more to the left so that as $\xi \to \infty$, u^*, as a first approximation, is uniform throughout the layer (except at the wall $\eta = 0$). The v^* profiles show that introduction of curvature changes the flat plate behaviour. The profiles peak in the middle of the layer and eventually die out as $\eta \to \infty$. The maximum value of v^* diminishes rapidly with increasing curvature so that again as $\xi \to \infty$, $v^* = 0$ to a first approximation.

The temperature profiles for Prandtl number of unity are expressed as $(1 - u^*)$ and for Prandtl number of 0·7 they are very much similar to $(1 - u^*)$ therefore in Fig. 6 we only plot the profiles for $\sigma = 10$.

It should be noted here that the heat flux results obtained from the finite difference solution agreed with (65) to at least three significant digits for $\sigma = 0·7$ and 1 and, to at least two significant digits for $\sigma = 10$.

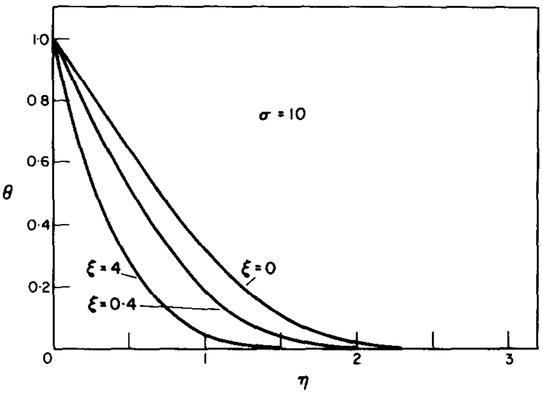

FIG. 6. Temperature profiles for Prandtl number 10.

REFERENCES

1. M. VAN DYKE, Higher approximations in boundary-layer theory, Part 1. General analysis, *J. Fluid Mech.* **14**, 161 (1962).
2. R. A. SEBAN and R. BOND, Skin-friction and heat-transfer characteristics of a laminar boundary layer on a cylinder in axial incompressible flow, *J. Aeronaut. Sci.* **18**, 671 (1951).
3. H. R. KELLY, A note on the laminar boundary layer on a circular cylinder in axial incompressible flow, *J. Aeronaut. Sci.* **21**, 634 (1954).
4. G. K. BATCHELOR, The skin friction on infinite cylinders moving parallel to their length, *Q. Jl. Mech. Appl. Math.* **7**, 179 (1954).
5. LORD RAYLEIGH, On the motion of solid bodies through viscous liquids, *Phil. Mag.* **21**(6), 697 (1911).
6. K. STEWARTSON, The asymptotic boundary layer on a circular cylinder in axial incompressible flow, *Q. Appl. Math.* **13**, 113 (1955).
7. M. B. GLAUERT and M. J. LIGHTHILL, The axisymmetric boundary layer on a long thin cylinder, *Proc. R. Soc.* **A230**, 188 (1955).

Résumé—On a analysé le problème de la couche limite laminaire et du transport de chaleur sur un long cylindre mince dans un écoulement uniforme. Des solutions ont été obtenues aussi bien pour de petites que pour de grandes valeurs du paramètre de courbure. Les solutions valables pour de petites valeurs de la courbure sont étendues pour s'appliquer également aux valeurs intermédiaires de la courbure. Ceci est obtenu en mettant les expressions sous forme de fractions. Finalement, ces fractions sont ajustées à la solution asymptotique afin de fournir des résultats de transport de chaleur et de frottement pour toutes les valeurs du paramètre de courbure. Les profils de vitesse et de température sont donnés sous forme de graphiques.

Zusammenfassung—Das Problem der laminaren Grenzschichtströmung und des Wärmeübergangs an einem langen dünnen Zylinder bei gleichmässiger Strömung wurde analysiert. Lösungen wurden sowohl für kleine als auch grosse Werte des Krümmungsparameters erhalten. Die für kleine Krümmungswerte geltenden Lösungen werden erweitert, um auch für mittlere Krümmungswerte anwendbar zu sein. Dies geschieht durch Aufgliederung der Ausdrücke in Teillösungen. Diese Teillösungen werden schliesslich mit der asymptotischen Lösung zusammengefasst, um Ergebnisse für den Wärmeübergang und die Wandreibung bei allen Werten des Krümmungsparameters zu liefern. Die Formen der Geschwindigkeits- und Temperaturprofile sind graphisch wiedergegeben.

Аннотация—Проанализирована задача течения и теплообмена ламинарного пограничного слоя при обтекании длинного тонкого цилиндра равномерным потоком. Получены решения для малых и больших значений параметра кривизны. Решения для малых значений кривизны распространены также на промежуточные значения. Для этого выражения разбиваются на части. Затем для этих частей составляется асимптотическое решение для получения данных по теплообмену и поверхностному трению для всех значений кривизны. Приводятся графики распределения скоростей и температур.